TEMPERATURE DISTRIBUTION IN A LIQUID SPHERICAL LAYER AROUND A SPHERE FALLING UNDER GRAVITY

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An examination is made of the stationary problem of temperature distribution in a liquid of known mass covering a solid sphere falling under gravity, the temperatures of both media with which the liquid is in contact being given. A solution has been obtained under the assumption of constant coefficient of thermal expansion of the liquid. A number of particular cases is examined, the question of stability of the liquid under the temperature distribution obtained is solved, and the total heat content of the liquid is found.

Let a liquid of mass m cover a solid sphere, r = = a, falling under gravity, the temperature T_e of the sphere and of the medium above the free surface of the liquid r = b (subscripts 1 and 2, respectively) being known. The assumption of a motionless liquid satisfies the stationary problem.

We write the heat conduction equation for the liquid with boundary conditions for temperature*

$$\frac{d}{dr}\left(\lambda r^2 \frac{dT}{dr}\right) = 0, \qquad \lambda r^2 \frac{dT}{dr}\bigg|_{r=a,b} = \alpha_1 a^2 \Delta T_1 = \alpha_2 b^2 \Delta T_2, \quad (1)$$

$$\left(\begin{matrix}\Delta T_1 = T_1 - T_{e1} \\ \Delta T_2 = T_{e2} - T_2\end{matrix}\right),$$

and the integral condition for mass

$$4\pi \int_a^b \frac{r^2}{v}\, dr = m \qquad (v \text{ — specific volume}). \quad (2)$$

The solution of the differential equation for constant thermal conductivity λ is

$$T = T_1 + \theta\,(1 - 1/\rho), \qquad 1 \leqslant \rho \leqslant \xi, \quad (3)$$

where ρ and ξ are the ratios of r and b to a. The constants of integration T_1 and θ, and also ξ (or b), are to be determined. Treating the α_i as constants, we use the boundary conditions to express the first two of the unknown quantities in terms of the third:

$$T_1 = \frac{\beta(\gamma T_{e1} + T_{e2})\xi^2 - \beta\gamma T_{e1}\xi + T_{e1}}{\beta(\gamma+1)\xi^2 - \beta\gamma\xi + 1}, \quad \theta = \frac{\beta\gamma\Delta T_e\, \xi^2}{\beta(\gamma+1)\xi^2 - \beta\gamma\xi + 1},$$

$$\beta = \alpha_2/\alpha_1, \qquad \gamma = \alpha_1 a/\lambda, \qquad \Delta T_e = T_{e2} - T_{e1}. \quad (4)$$

We shall restrict the examination to an incompressible liquid with constant thermal expansion coefficient δ

$$v = v_0 e^{\delta T}, \qquad v_0 = \text{const}, \quad (5)$$

for which (2) takes the form

$$\vartheta^3 e^{-\vartheta}[\mathrm{Ei}(\vartheta) - \mathrm{Ei}(\vartheta/\xi)] + (3\vartheta^2 + 3\vartheta\xi + 2\xi^2)\xi \exp[-(1 - 1/\xi)\vartheta] - (3\vartheta^2 + 3\vartheta + 2) = ne^{\delta T_1}, \quad (6)$$

$$\left(\vartheta = \delta\theta, \qquad n = \frac{3}{2\pi}\frac{mv_0}{a^3} < \frac{1}{2}\right),$$

which is transcendental relative to ξ. After this is solved, T_1 and θ are determined from (4). In what follows we exclude the trivial special cases when

$$\alpha_1 = 0 \quad (T = T_{e2}), \quad \alpha_2 = 0 \quad (T = T_{e1}), \quad \Delta T_e = 0 \quad (T = T_e),$$

for which

$$\xi = \sqrt[3]{{}^1/{}_2 n\, e^{\delta T} + 1}.$$

We note that

$$\frac{1}{\Delta T_e}\{\Delta T, \Delta T_1, \Delta T_2, \theta\} < 1 \qquad (\Delta T = T_2 - T_1). \quad (7)$$

When $\delta = 0$, equation (6) gives

$$\xi = \sqrt[3]{{}^1/{}_2 n + 1}. \quad (8)$$

When $\varepsilon = \delta\Delta T_e$ is small, viz. $|\varepsilon| \ll \min(1; 1/\gamma)$, we obtain from (6)

$$[\beta(\gamma+1)\xi^2 - \beta\gamma\xi + 1][2(\xi^3 - 1) - ne^{\delta T_{e1}}] = \beta\xi^2[\gamma(2\xi^3 - 5\xi^2 + 3) + ne^{\delta T_{e1}}]\varepsilon. \quad (9)$$

The root desired is close to

$$\xi = \sqrt[3]{{}^1/{}_2 ne^{\delta T_{e1}} + 1}. \quad (10)$$

For example, when $ne^{\delta T_{e1}} \gg 2\max(6, \beta^{-3/2}, \gamma^{-3/2})$, we have

$$\xi \approx \left(1 + \frac{\gamma}{\gamma+1}\frac{\varepsilon}{3}\right)\left(\frac{n}{2}\right)^{1/3} \exp\left(\frac{\delta T_{e1}}{3}\right). \quad (11)$$

For n sufficiently small, (6) gives

$$\xi = 1 + n\left[\beta\frac{\beta\gamma - 2}{(1+\beta)^2}\varepsilon n + 2(3 + 2\vartheta_0 - 3\vartheta_0^3)\exp\left(-\delta\frac{T_{e1} + \beta T_{e2}}{1+\beta}\right)\right]^{-1}. \quad (12)$$

The necessary condition for the existence of a solution of (12) will be

$$0 < \xi - 1 \leqslant \min\left[1, \frac{\beta+1}{\beta}\frac{1}{\gamma+2}, \frac{1}{|\vartheta_0|}, \frac{(\beta+1)^2}{\beta|2 - \beta\gamma||\varepsilon|}, \frac{1}{\beta}\frac{T_{e1} + \beta T_{e2}}{\gamma T_{e1} + 2T_{e2}}\right] \qquad \left(\vartheta_0 = \frac{\beta\gamma\varepsilon}{\beta+1}\right). \quad (13)$$

*What is intended here is to determine the heat transfer coefficient α, in the sense of contact heat conduction (see, e.g., [1]), as distinct from the convective heat transfer coefficient ordinarily used in technical calculations [2].

On the other hand, when $ne^{\delta T_{e_2}} \gg 2 \max(1; \beta^{-2/3}; |\vartheta_1|^3)$ we obtain from (6)

$$2\xi^3 + \vartheta_1^3[\ln\xi + \mathrm{Ei}(\vartheta_1) - \ln|\vartheta_1| - C] - (3\vartheta_1^2 + 3\vartheta_1 + 1)e^{\vartheta_1} = \\ = ne^{\delta T_{e2}}\left[1 + \frac{\vartheta_1}{(\gamma+1)\xi}\right] \qquad \left(\vartheta_1 = \frac{\gamma\varepsilon}{\gamma+1}\right), \tag{14}$$

where C is the Euler constant. The root is close to

$$\xi = \sqrt[3]{{}^1/_2\, n} \exp {}^1/_3\, \delta T_{e2}\ . \tag{15}$$

Let us now elucidate the question of hydrodynamical stability of the liquid for the temperature distribution (3) obtained. The condition for absence of convection is [3]

$$\rho^2 \frac{d}{d\rho} \ln T > -\frac{\delta}{c}k, \quad \text{or (in the case given)} \quad \frac{\theta}{T} > -\frac{\delta}{c}k, \tag{16}$$

where c is the specific heat of the liquid, and k is the product of the gravitational constant and the mass of the solid sphere divided by its radius. When $\Delta T_e > 0$, inequality (16) is satisfied, and there is stability. When $\Delta T_e < 0$ we obtain

$$\frac{T_2}{|\theta|} > \frac{c}{\delta k},$$

$$\text{or} \quad \left(\frac{\gamma+1}{\gamma}T_{e2} - \frac{c}{\delta k}|\Delta T_e|\right)\xi^2 - T_{e2}\xi + \frac{T_{e1}}{\beta\gamma} > 0. \tag{17}$$

If a root of (6) does not satisfy (17), convection is possible in the liquid. When this occurs, the temperature distribution generally becomes different from (3).

The total heat content of the liquid is

$$J = 4\pi c \int_1^\xi \frac{T}{v}\rho^2 d\rho = \frac{2\pi a^3}{\delta v_0} e^{-\delta T_1}\Big\{(\delta T_1 + \vartheta - 3)\vartheta^3 e^{-\vartheta}\Big[\mathrm{Ei}(\vartheta) - \\ - \mathrm{Ei}\left(\frac{\vartheta}{\xi}\right)\Big] + [(\delta T_1 + \vartheta)(3\vartheta^2 + 3\vartheta\xi + 2\xi^2) - 3\vartheta(\vartheta + \xi)]\xi \times \\ \times \exp\left(-\frac{\xi - 1}{\xi}\vartheta\right) - [(\delta T_1 + \vartheta)(3\vartheta^2 + 3\vartheta + 2) - 3\vartheta(\vartheta + 1)]\Big\}, \tag{18}$$

$$J = cm\left[T_1 + \theta - \frac{3\theta}{n}(\xi^2 - 1)\right] \quad \text{when } \delta = 0, \tag{19}$$

where ξ is determined from (8).

REFERENCES

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